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SCATTERING MATRICES FOR IONOSPHERE MODELS

by

ARNOLD RUSSEK

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ABSTRACT

This paper represents an attempt to obtain useable approximations, for two special cases, to the scattering matrix for an inhomogeneous, anisotropic medium of finite thickness bounded above and below by semi-infinite homogeneous, isotropic media. The ultimate purpose of this attack will be to obtain the transmission and reflection coefficients for a plane wave incident at an arbitrary angle on an ionospheric layer with constant ϵ and μ in a uniform earth's magnetic field as a functional of both the charge distribution function $N(z)$ and the collision frequency $\nu(z)$.

First, the scattering matrix for an anisotropic multilayer is obtained. This will be a good approximation to the continuously varying medium at low frequencies where each layer is but a fraction of a wavelength in thickness.

In the high frequency case, where a solution in series form has already been obtained by Bremmer, it is found that a considerable simplification can be effected by expanding the conductivity matrix in inverse powers of frequency.

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1. INTRODUCTION

The use of electromagnetic waves for determining the structure and properties of the ionosphere has stimulated interest in investigating the behavior of these waves in a horizontally stratified ionized medium in the presence of a fixed external magnetic field. Such a medium is a reasonable approximation to the ionosphere. The problem is most readily treated in the range of high frequencies, where recourse can be had to various modifications and improvements of the W.K.B. approximations or, alternatively, to the series expansion technique. The lowest order of approximation in the latter case is the geometrical optics approximation [1].

When the frequency of the incident electromagnetic wave is extremely low (below 100 K.C.), neither the series expansion technique nor the W.K.B. approximations are applicable. In this frequency range, the anisotropic multilayer solution of Section 2 may prove useful.

This method consists of approximating the ionosphere by a piecewise constant medium composed of a number of homogeneous, anisotropic slabs, each of which is but a fraction of a wavelength in thickness. The case of an isotropic multilayer has been treated by many writers [2] and the method used here is basically a generalization of the procedure used in that case.

For a given incident plane wave, the electromagnetic field in each slab is known to consist of a linear combination of four elementary plane waves, two upgoing and two downcoming. The coefficients of these elementary plane waves are related by the continuity conditions on the electric and magnetic field across every interface separating a pair of slabs. In this manner, it is possible, by ordinary algebraic manipulation to determine the amplitudes of the outgoing plane waves in terms of the amplitudes of the incoming plane waves. This connection is given by a matrix known as the "scattering matrix".

As the number of slabs in the multilayer is made infinite, the thickness of each slab going to zero, the multilayer becomes, in the limit, a continuous medium. The multilayer solution can be shown to reduce, in this limit, to an iterate solution obtained by Bremmer [3] for the electromagnetic field in a continuously

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[1] See, for example, H.G. Booker, Phil. Trans. Roy. Soc. 237, 411, 1938

[2] See, for example, R.K. Luneberg, The Propagation of Electromagnetic Plane Waves in Plane Parallel Layers. Research Report No.172-3,W.S.C. Math.Res.Grp. 1947

[3] H.Bremmer, Terrestrial Radio Waves, pp.301-336 Elsevier Publishing Co., N.Y., 1950.

varying anisotropic medium. While the latter solution is exact, it is in series form, and the series is rapidly convergent only at high frequencies. For practical purposes it can therefore be used only in this range. The multilayer solution, on the other hand, is applicable only in the low frequency range, since each layer of the multilayer must be considerably smaller than a wavelength in thickness and, for short wavelengths, the number of layers would have to be too large for practical computation.

In Section 3 a simple formula is derived for the first few terms in a series expression for the scattering matrix as explicit functions of the ionospheric parameters for the case in which the frequency of the incident wave is greater than both the maximum critical frequency and the gyromagnetic frequency of the ionospheric layer. The starting point of this discussion is the Bremmer solution, which, although rapidly convergent, is given in terms of complicated functions of the solutions of a quartic equation. By expanding the quantities appearing in the wave equation for the electric field in inverse powers of the frequency of the incident wave, a technique used by Ginsburg [4], the algebraic complexity of the Bremmer solution is greatly simplified.

The simultaneous use of the series expansion technique and the iterated solution technique has a further substantial advantage. The formula obtained includes the effect of the collision frequency on the electromagnetic field explicitly, whereas this is ordinarily too complicated to be handled by either the iterated solution technique or the series expansion technique alone.

The technique used in section 3 to treat the high frequency case may also, with suitable modifications, be used for the case of low charge density. This problem will form the subject of another report. Only with this approximation, along with the two approximations treated in this report can numerical computations on actual ionospheric models effectively be performed. For this reason no numerical work is included in this paper.

[4] V. L. Ginsburg, Journal of Physics, Vol. VII No. 6, 289 (1943)

2. THE GENERAL ANISOTROPIC MULTILAYER

In this section we will consider an obliquely incident plane wave in a stratified anisotropic medium which consists of N + 2 homogeneous strata. The zeroth stratum, which extends from $z = -\infty$ to $z = 0$ and the $N + 1^{st}$ stratum, which extends from $z = L$ to $z = +\infty$ will both be assumed to be isotropic; the other N strata will be restricted only in that the permeability $\mu^{(\upsilon)}(\upsilon=1,\ldots,N)$ will be required to be scalar. Hence $\bar{\bar{\epsilon}}^{(\upsilon)}$ and $\bar{\bar{\sigma}}^{(\upsilon)}$ may be taken to be arbitrary tensor quantities, although the most useful case will be that in which $\mu^{(\upsilon)}$ and $\epsilon^{(\upsilon)}$ are both scalars and the conductivity tensor $\bar{\bar{\sigma}}^{(\upsilon)}$ is that resulting from a free charge distribution in a uniform earth's magnetic field. If N is then taken sufficiently large, this special case will yield a good approximation to the physical situation of a plane wave obliquely incident on the ionosphere.

For the sake of simplicity, the x- and y- coordinate axes will be so chosen that the propagation vector in the zeroth medium (and hence in all media) lies in the x-z plane, the z-axis having already been chosen perpendicular to the strata.

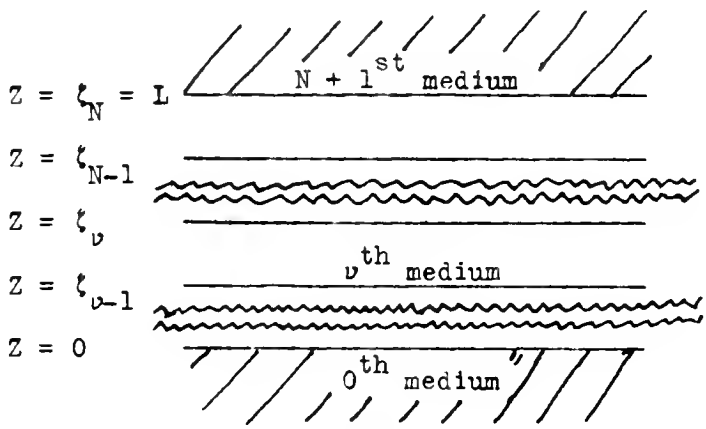

Fig. 1

Under these conditions, the electric field $\bar{E}^{(\upsilon)}$ in the υ^{th} medium can be expressed as a sum of the form*

$$(1)\quad \bar{E}^{(\upsilon)}(x,z,t) = \sum_{n=1}^{4} C_n^{(\upsilon)} \bar{A}_n^{(\upsilon)} e^{i\left[-\omega t + kx + h_n^{(\upsilon)} z\right]}$$

where $C_n^{(\upsilon)}$ is a complex amplitude, $\bar{A}_n^{(\upsilon)}$ is a complex polarization vector which depends only on the properties of the medium as does the z-component of

---

* The justification for assuming an expansion of the form (1) for $\bar{E}$ will be given in Section 2.A. It is also discussed in H. Bremmer, Terrestrial Radio Waves, pp. 310-316.

the propagation vector $h_n^{(\nu)}$. The value of the x-component of the propagation vector k is known to be independent of the medium and depends only upon the direction of incidence and the frequency of the incident wave.

Our first problem (section 1A) will be to determine $\bar{A}_n^{(\nu)}$ and $h_n^{(\nu)}$ in each of the media as a function of the physical properties of that medium. Then, in section 1B, by applying continuity conditions across the boundary separating the ν^{th} medium from the $\nu\text{-}1^{st}$, we shall be able to determine the amplitudes $C_n^{(\nu)}$ as a linear combination of the amplitudes $C_n^{(\nu-1)}$ in the slab below. By repeating this process, we can determine the amplitudes in any slab. Then, using equation (1), we can specify the electric field at any point in the medium. From this, the values of the components of the magnetic field can be computed from Maxwell's equations, if desired.

A - Determination of the Propagation and Polarization Vectors.

If we assume $\overline{\overline{\mu}}$ and $\overline{\overline{\epsilon}}$ to be arbitrary tensors which do not vary with time and if we assume that $\overline{E}$, $\overline{H}$ and $\overline{j}$ have the harmonic time dependence $e^{-i\omega t}$, we can write Maxwell's equations as

$$(2) \qquad \nabla \times \overline{E} = +i\omega \overline{\overline{\mu} H}$$
$$\nabla \times \overline{H} = -i\omega \overline{\overline{\epsilon} E} + \overline{j}$$

where $\overline{\overline{\mu} H}$ and $\overline{\overline{\epsilon} E}$ are the vectors resulting from the matrix multiplication of $\overline{\overline{\mu}}$ and $\overline{H}$ and $\overline{\overline{\epsilon}}$ and $\overline{E}$ respectively. If, now, we restrict $\overline{\overline{\mu}}$ to be a scalar quantity designated, simply, by μ, we can eliminate $\overline{H}$ from (2) to get

$$(3) \qquad \mu \nabla \times \frac{1}{\mu} \nabla \times \overline{E} = \omega^2 \mu \, \overline{\overline{\epsilon} E} + i \omega \mu \overline{j} \, .$$

We now replace $\overline{j}$ by $\overline{\overline{\sigma} E}$, where $\overline{\overline{\sigma}}$ is an arbitrary conductivity tensor and rewrite (3) as

$$(4a) \qquad \mu \nabla \times \frac{1}{\mu} \nabla \times \overline{E} = \omega^2 \mu \left(\overline{\overline{\epsilon} E} + \frac{i}{\omega} \overline{\overline{\sigma} E} \right) = \overline{\overline{\rho} E}$$

where we have abbreviated the right hand side by writing

$$(4b) \qquad \overline{\overline{\rho}} = \omega^2 \mu \left(\overline{\overline{\epsilon}} + \frac{i}{\omega} \overline{\overline{\sigma}} \right) .$$

Now, in each of the media, μ is constant, so that the left hand side of (4a) becomes $\nabla \times \nabla \times \overline{E}$. Substituting for $\overline{E}$ a plane wave solution

$$\overline{E} = \overline{A} \, e^{i \left[-\omega t + kx + hz \right]}$$

into (4), we get, writing out the equations fully,

$$(5) \qquad \begin{aligned} A_x (h^2 - \rho_{xx}^{(v)}) + A_y (-\rho_{xy}^{(v)}) + A_z (-hk - \rho_{xz}^{(v)}) &= 0 \\ A_x (-\rho_{yx}^{(v)}) + A_y (h^2 + k^2 - \rho_{yy}^{(v)}) + A_z (-\rho_{yz}^{(v)}) &= 0 \\ A_x (-hk - \rho_{zx}^{(v)}) + A_y (-\rho_{zy}^{(v)}) + A_z (k^2 - \rho_{zz}^{(v)}) &= 0 \end{aligned}$$

The set of linear equations (5), being homogeneous, will possess a non-trivial solution if and only if the determinant of the coefficients vanishes. Setting

this determinant equal to zero yields the following quartic equation in h for a fixed value of k.

$$
\begin{aligned}
(6)\quad & -h^4\rho_{zz}^{(\upsilon)} - h^3\left\{k(\rho_{zx}^{(\upsilon)} + \rho_{xz}^{(\upsilon)})\right\} + h^2\left\{\rho_{zz}^{(\upsilon)}(\rho_{xx}^{(\upsilon)} + \rho_{yy}^{(\upsilon)}) - k^2(\rho_{xx}^{(\upsilon)} + \rho_{zz}^{(\upsilon)})\right. \\
& \left. -\rho_{xz}^{(\upsilon)}\rho_{zx}^{(\upsilon)} - \rho_{yz}^{(\upsilon)}\rho_{zy}^{(\upsilon)}\right\} - h\left\{k^3(\rho_{zx}^{(\upsilon)} + \rho_{xz}^{(\upsilon)}) + k\left[\rho_{yx}^{(\upsilon)}\rho_{zy}^{(\upsilon)} + \rho_{xy}^{(\upsilon)}\rho_{yz}^{(\upsilon)}\right.\right. \\
& \left.\left. -\rho_{yy}^{(\upsilon)}(\rho_{zx}^{(\upsilon)}+\rho_{xz}^{(\upsilon)})\right]\right\} + \left\{-k^4\rho_{xx}^{(\upsilon)}+k^2\left[\rho_{xx}^{(\upsilon)}(\rho_{yy}^{(\upsilon)}+\rho_{zz}^{(\upsilon)}) - \rho_{xz}^{(\upsilon)}\rho_{zx}^{(\upsilon)} - \rho_{yx}^{(\upsilon)}\rho_{xy}^{(\upsilon)}\right]\right. \\
& \left. - \begin{vmatrix} \rho_{xx}^{(\upsilon)} & \rho_{xy}^{(\upsilon)} & \rho_{xz}^{(\upsilon)} \\ \rho_{yx}^{(\upsilon)} & \rho_{yy}^{(\upsilon)} & \rho_{yz}^{(\upsilon)} \\ \rho_{zx}^{(\upsilon)} & \rho_{zy}^{(\upsilon)} & \rho_{zz}^{(\upsilon)} \end{vmatrix} \right\} = 0
\end{aligned}
$$

Thus, there will, in general, be four values of h in the υ^{th} layer for which plane wave solutions of (4) with a given value of k exist. We will designate them by $h_n^{(\upsilon)}$ (n=1,2,3,4) and the corresponding polarization vectors, obtained by substituting these values for $h_n^{(\upsilon)}$ into (5), by $\overline{A}_n^{(\upsilon)}$. The most general plane wave solution of the wave equation in the υ^{th} medium will therefore be a linear combination of the four elementary solutions we have just obtained multiplied by four amplitude coefficients $C_n^{(\upsilon)}$, thereby justifying the form for $\overline{E}$ assumed in equation (1).

As stated before, the polarization vectors $\overline{A}_n^{(\upsilon)}$ can be determined (to within an arbitrary normalization factor) by solving equations (5); that is, the ratio of any two components of $\overline{A}_n^{(\upsilon)}$ can be determined from (5). Thus, we have

$$
\begin{aligned}
(7)\quad A_{nx}^{(\upsilon)} &= m_n^{(\upsilon)}\left\{\rho_{xy}^{(\upsilon)}\rho_{yz}^{(\upsilon)} + \left[h_n^{(\upsilon)}k + \rho_{xz}^{(\upsilon)}\right]\left[(h_n^{(\upsilon)})^2 + k^2 - \rho_{yy}^{(\upsilon)}\right]\right\} \\
A_{ny}^{(\upsilon)} &= m_n^{(\upsilon)}\left\{\left[h_n^{(\upsilon)}k + \rho_{xz}^{(\upsilon)}\right]\rho_{yx}^{(\upsilon)} + \rho_{yz}^{(\upsilon)}\left[(h_n^{(\upsilon)})^2 - \rho_{xx}^{(\upsilon)}\right]\right\} \\
A_{nz}^{(\upsilon)} &= m_n^{(\upsilon)}\left\{\left[(h_n^{(\upsilon)})^2 + k^2 - \rho_{yy}^{(\upsilon)}\right]\left[(h_n^{(\upsilon)})^2 - \rho_{xx}^{(\upsilon)}\right] - \rho_{xy}^{(\upsilon)}\rho_{yx}^{(\upsilon)}\right\}
\end{aligned}
$$

where $m_n^{(\nu)}$ is an arbitrary normalization factor.

It is interesting to note that the quartic equation (6) for $h_n^{(\nu)}$ reduces to a bi-quadratic if either

a) $k = 0$ (vertical incidence), or

b) $\rho_{zx}^{(\nu)} + \rho_{xz}^{(\nu)} = 0$ and $\rho_{yx}^{(\nu)}\rho_{zy}^{(\nu)} + \rho_{xy}^{(\nu)}\rho_{yx}^{(\nu)} = 0$.

When this happens, we have $h_1^{(\nu)} = - h_3^{(\nu)}$ and $h_2^{(\nu)} = -h_4^{(\nu)}$ and many simplifications in algebraic tedium result.

B- Continuity Conditions and the Coupling Coefficients

The continuity conditions across a boundary require that the tangential components of both $\overline{E}$ and $\overline{H}$ must be continuous across a boundary which, in this case, is taken to be the plane $z = \zeta_\nu$ which separates the ν^{th} medium from the $(\nu-1)^{th}$ medium. The continuity of the tangential components of $\overline{E}$ yields

$$(8)\quad a)\quad E_x^{(\nu-1)}(\zeta_{\nu-1}) = E_x^{(\nu)}(\zeta_{\nu-1})$$

$$b)\quad E_y^{(\nu-1)}(\zeta_{\nu-1}) = E_y^{(\nu)}(\zeta_{\nu-1})$$

where we have suppressed the dependence of $\overline{E}$ on x and t. It is understood that (8) is to hold for all values of x and y at all times. We are now in a position to justify our assumption that the x-component of the propagation vector is the same for all media. $\overline{E}$ is a function of x only through the exponential factor e^{ikx}. Hence (8) states that $e^{ik^{(\nu)}x} = e^{ik^{(\nu-1)}x}$ from which we see that $k^{(\nu)} = k^{(\nu-1)} = \cdots = k^{(o)} \equiv k$ since k is defined as the value of the x-component of the propagation vector in the zeroth medium.

Because we are dealing only with the electric field, the two continuity conditions on $\overline{H}$ must be changed to additional conditions on $\overline{E}$ by means of Maxwell's equations. From

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = +i\omega\mu H_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = +i\omega\mu H_y$$

we get, realizing that the electric field has no y dependence and an x dependence equal to e^{ikx}

$$(9)\quad a)\quad \frac{1}{\mu^{(\nu-1)}} \left.\frac{\partial E_y^{(\nu-1)}}{\partial z}\right|_{\zeta_{\nu-1}} = \frac{1}{\mu^{(\nu)}} \left.\frac{\partial E_y^{(\nu)}}{\partial z}\right|_{\zeta_{\nu-1}}$$

$$b)\quad \frac{1}{\mu^{(\nu-1)}}\left[\left.\frac{\partial E_x^{(\nu-1)}}{\partial z}\right|_{\zeta_{\nu-1}} - ikE_z^{(\nu-1)}(\zeta_{\nu-1})\right] = \frac{1}{\mu^{(\nu)}}\left[\left.\frac{\partial E_x^{(\nu)}}{\partial z}\right|_{\zeta_{\nu-1}} - ikE_z^{(\nu)}(\zeta_{\nu-1})\right]$$

Our next task is to substitute (1) into equations (8) and (9) and thus finally express the continuity conditions on the electromagnetic field as a set of linear relations which give the amplitudes $C_n^{(\nu)}$ in terms of the amplitudes $C_n^{(\nu-1)}$. The aforementioned substitution yields

$$\text{a)} \quad \sum_{n=1}^{4} C_n^{(\nu-1)} A_{nx}^{(\nu-1)} e^{ih_n^{(\nu-1)}\zeta_{\nu-1}} = \sum_{n=1}^{4} C_n^{(\nu)} A_{nx}^{(\nu)} e^{ih_n^{(\nu)}\zeta_{\nu-1}}$$

$$\text{(10) b)} \quad \sum_{n=1}^{4} C_n^{(\nu-1)} A_{ny}^{(\nu-1)} e^{ih_n^{(\nu-1)}\zeta_{\nu-1}} = \sum_{n=1}^{4} C_n^{(\nu)} A_{ny}^{(\nu)} e^{ih_n^{(\nu)}\zeta_{\nu-1}}$$

$$\text{c)} \quad \frac{1}{\mu^{(\nu-1)}} \sum_{n=1}^{4} C_n^{(\nu-1)} A_{ny}^{(\nu-1)} ih_n^{(\nu-1)} e^{ih_n^{(\nu-1)}\zeta_{\nu-1}} = \frac{1}{\mu^{(\nu)}} \sum_{n=1}^{4} C_n^{(\nu)} A_{ny}^{(\nu)} ih_n^{(\nu)} e^{ih_n^{(\nu)}\zeta_{\nu-1}}$$

$$\text{d)} \quad \frac{1}{\mu^{(\nu-1)}} \sum_{n=1}^{4} C_n^{(\nu-1)} B_n^{(\nu-1)} e^{ih_n^{(\nu-1)}\zeta_{\nu-1}} = \frac{1}{\mu^{(\nu)}} \sum_{n=1}^{4} C_n^{(\nu)} B_n^{(\nu)} e^{ih_n^{(\nu)}\zeta_{\nu-1}}$$

where we have defined $B_n^{(\nu)}$ as

$$\text{(11)} \quad B_n^{(\nu)} = h_n^{(\nu)} A_{nx}^{(\nu)} - kA_{nz}^{(\nu)} \quad .$$

It should be noted that in (10) the exponential $e^{i[-\omega t+kx]}$, which appears on both sides, has been cancelled. The solutions of equation (10) are, in matrix form,

$$\text{(12)} \quad \bar{C}^{(\nu)} = \bar{\bar{T}}^{(\nu)} \bar{C}^{(\nu-1)}$$

where $\bar{C}^{(\nu)}$ and $\bar{C}^{(\nu-1)}$ are understood to be the four element column vectors $C_n^{(\nu)}$ and $C_n^{(\nu-1)}$. The components of $\bar{\bar{T}}^{(\nu)}$ are given by

$$\text{(13)} \quad T_{nm}^{(\nu)} = e^{i\left[h_m^{(\nu-1)} - h_n^{(\nu)}\right]\zeta_{\nu-1}} \frac{\Delta_{nm}^{(\nu)}}{\Delta^{(\nu)}} \quad .$$

where

$$(14a)\quad \Delta^{(\nu)} = \begin{vmatrix} A_{1x}^{(\nu)} , & A_{2x}^{(\nu)} , & A_{3x}^{(\nu)} , & A_{4x}^{(\nu)} \\ A_{1y}^{(\nu)} , & A_{2y}^{(\nu)} , & A_{3y}^{(\nu)} , & A_{4y}^{(\nu)} \\ h_1^{(\nu)}A_{1y}^{(\nu)} , & h_2^{(\nu)}A_{2y}^{(\nu)} , & h_3^{(\nu)}A_{3y}^{(\nu)} , & h_4^{(\nu)}A_{4y}^{(\nu)} \\ B_1^{(\nu)} , & B_2^{(\nu)} , & B_3^{(\nu)} , & B_4^{(\nu)} \end{vmatrix}$$

and $\Delta_{nm}^{(\nu)}$ is $\Delta^{(\nu)}$ with the n^{th} column removed and replaced by the column

$$A_{mx}^{(\nu-1)} , A_{my}^{(\nu-1)} , \frac{\mu^{(\nu)}}{\mu^{(\nu-1)}} h_m^{(\nu-1)}A_{my}^{(\nu-1)} , \frac{\mu^{(\nu)}}{\mu^{(\nu-1)}} B_m^{(\nu-1)} .$$

involving the properties of the $\nu\text{-}1^{st}$ medium. Thus, for example,

$$(14b)\ \Delta_{2m}^{(\nu)} = \begin{vmatrix} A_{1x}^{(\nu)} & A_{mx}^{(\nu-1)} & A_{3x}^{(\nu)} & A_{4x}^{(\nu)} \\ A_{1y}^{(\nu)} & A_{my}^{(\nu-1)} & A_{3y}^{(\nu)} & A_{4y}^{(\nu)} \\ h_1^{(\nu)}A_{1y}^{(\nu)} , & \frac{\mu^{(\nu)}}{\mu^{(\nu-1)}} h_m^{(\nu-1)}A_{my}^{(\nu-1)} , & h_3^{(\nu)}A_{3y}^{(\nu)} , & h_4^{(\nu)}A_{4y}^{(\nu)} \\ B_1^{(\nu)} & \frac{\mu^{(\nu)}}{\mu^{(\nu-1)}} B_m^{(\nu-1)} & B_3^{(\nu)} & B_4^{(\nu)} \end{vmatrix}$$

C - The Scattering Matrix

We may now formally carry out the last step in our program; namely obtaining $C_n^{(\nu)}$ in terms of the $C_n^{(o)}$. The connection, in matrix form, is simply given by

$$(15) \quad \overline{C}^{(\nu)} = \overline{\overline{T}}^{(\nu)}\, \overline{\overline{T}}^{(\nu-1)} \cdots \overline{\overline{T}}^{(2)}\, \overline{\overline{T}}^{(1)}\, \overline{C}^{(o)} = \prod_{r=1}^{\nu} \overline{\overline{T}}^{(r)}\, \overline{C}^{(o)}.$$

Of particular interest are the amplitudes in the $N + 1^{st}$ medium. If we substitute $N + 1$ for ν in (15), we will get the amplitudes of the four polarizations in the homogeneous isotropic medium above the anisotropic multilayer in terms of the amplitudes of the four polarizations in the homogeneous isotropic medium below. Because of the special nature of these two media, the four polarizations wil be degenerate. That is, equations (7) will break down and yield $A_{nx} = A_{ny} = A_{nz} = m_n \cdot 0$ so that the ratios of the various pairs of components of the polarization vectors are not uniquely determined, as is well known. Hence we need only define any complete set of polarizations in these media such as, for instance, two oppositely directed circular polarizations or two perpendicular linear polarizations for both the upgoing and downcoming waves.

Defining $\overline{\overline{T}}$, the total connection, or coupling, matrix for the entire multilayer by

$$(16) \quad \overline{\overline{T}} = \prod_{\nu=1}^{N+1} \overline{\overline{T}}^{(\nu)} = \overline{\overline{T}}^{(N+1)}\, \overline{\overline{T}}^{(N)} \cdots \overline{\overline{T}}^{(2)}\, \overline{\overline{T}}^{(1)} ,$$

where care must be taken to preserve the order of the factors, equation (15) becomes, finally,

$$(17) \quad C_n^{(N+1)} = \sum_{r=1}^{4} T_{nr} C_r^{(o)}, \qquad n = 1,2,3,4.$$

Equation (17) determines the electric field above the multilayer in terms of the electric field below the multilayer by virtue of equation (1) (if we define any complete set of polarization vectors in these two isotropic media). Usually, however, in ionospheric applications, the amplitudes of the two downcoming waves in the zeroth medium and of the two upgoing waves in the $N + 1^{st}$ medium will be the unknowns, to be determined

in terms of the upgoing waves in the zeroth medium and the downcoming waves in the $N + 1^{st}$ medium. The amplitudes of the downcoming waves in the $N + 1^{st}$ medium are usually equal to zero, for in most cases, there is no wave incident on the ionosphere from above. At any rate, it is desirable to know the amplitudes of the outgoing waves in terms of the amplitudes of the incoming waves. The matrix which gives this connection is known as the scattering matrix and is generally denoted by $\bar{\bar{S}}$. Mathematically, $\bar{\bar{S}}$ is defined by

$$\begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \\ c_3^{(N+1)} \\ c_4^{(N+1)} \end{pmatrix} = \bar{\bar{S}} \begin{pmatrix} c_3^{(0)} \\ c_4^{(0)} \\ c_1^{(N+1)} \\ c_2^{(N+1)} \end{pmatrix} \tag{18}$$

where we have numbered the modes so that the subscripts 1 and 2 refer to downcoming waves while the subscripts 3 and 4 refer to upgoing waves.

It would be well to mention that in the case of the multilayer, the distinction between the ordinary and extraordinary rays in each layer is purely arbitrary. It is only in the continuously varying anisotropic medium at relatively high frequencies that the concept of a ray is at all sensible; hence the notation of ordinary and extraordinary rays cannot be applied to the multilayer to advantage.

It remains now only to give the components of the scattering matrix $\bar{\bar{S}}$ in terms of the connection, or coupling, matrix $\bar{\bar{T}}$. This can be done most simply by solving the four linear simultaneous equations given by (17) for $c_1^{(0)}, c_2^{(0)}, c_3^{(N+1)}, c_4^{(N+1)}$ in terms of $c_3^{(0)}, c_4^{(0)}, c_1^{(N+1)}, c_2^{(N+1)}$. Now, $\bar{c}^{(N+1)} = \bar{\bar{T}}\, \bar{c}^{(0)}$, or in partially expanded form,

$$\begin{pmatrix} c_1^{(N+1)} \\ c_2^{(N+1)} \end{pmatrix} = \bar{\bar{t}}_a \begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \end{pmatrix} + \bar{\bar{t}}_b \begin{pmatrix} c_3^{(0)} \\ c_4^{(0)} \end{pmatrix}$$

$$\begin{pmatrix} c_3^{(N+1)} \\ c_4^{(N+1)} \end{pmatrix} = \bar{\bar{t}}_c \begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \end{pmatrix} + \bar{\bar{t}}_d \begin{pmatrix} c_3^{(0)} \\ c_4^{(0)} \end{pmatrix} \tag{19a}$$

where $\bar{\bar{t}}_a$, $\bar{\bar{t}}_b$, $\bar{\bar{t}}_c$, $\bar{\bar{t}}_d$ are the two-by-two matrices

$$(19b)\qquad \bar{\bar{t}}_a = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \qquad \bar{\bar{t}}_b = \begin{pmatrix} T_{13} & T_{14} \\ T_{23} & T_{24} \end{pmatrix}$$

$$\bar{\bar{t}}_c = \begin{pmatrix} T_{31} & T_{32} \\ T_{41} & T_{42} \end{pmatrix} \qquad \bar{\bar{t}}_d = \begin{pmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{pmatrix}$$

Solving (19) for the outgoing waves in terms of the incoming ones yields the following expression for $\bar{\bar{S}}$

$$(20)\begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \\ c_3^{(N+1)} \\ c_4^{(N+1)} \end{pmatrix} = \bar{\bar{S}} \begin{pmatrix} c_3^{(0)} \\ c_4^{(0)} \\ c_1^{(N+1)} \\ c_2^{(N+1)} \end{pmatrix} = \begin{pmatrix} -\bar{\bar{t}}_a^{-1}\bar{\bar{t}}_b & , & \bar{\bar{t}}_a^{-1} \\ \bar{\bar{t}}_d - \bar{\bar{t}}_c\bar{\bar{t}}_a^{-1}\bar{\bar{t}}_b & , & \bar{\bar{t}}_c\bar{\bar{t}}_a^{-1} \end{pmatrix} \begin{pmatrix} c_3^{(0)} \\ c_4^{(0)} \\ c_1^{(N+1)} \\ c_2^{(N+1)} \end{pmatrix}$$

if $\bar{\bar{t}}_a$ is non-singular.

Taking the amplitudes $c_1^{(N+1)}$ and $c_2^{(N+1)}$ of the downcoming waves in the $N+1^{st}$ medium to be zero (which usually is the case), equation (20) simplifies to

$$(21a)\qquad \begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \end{pmatrix} = -\bar{\bar{t}}_a^{-1}\bar{\bar{t}}_b \begin{pmatrix} c_3^{(0)} \\ c_4^{(0)} \end{pmatrix}$$

$$\begin{pmatrix} c_3^{(N+1)} \\ c_4^{(N+1)} \end{pmatrix} = (\bar{\bar{t}}_d - \bar{\bar{t}}_c\bar{\bar{t}}_a^{-1}\bar{\bar{t}}_b) \begin{pmatrix} c_3^{(0)} \\ c_4^{(0)} \end{pmatrix}$$

which gives the amplitudes $c_1^{(0)}$ and $c_2^{(0)}$ of the reflected waves, and the amplitudes $c_3^{(N+1)}$ and $c_4^{(N+1)}$ of the transmitted waves in terms of the amplitudes $c_3^{(0)}$ and $c_4^{(0)}$ of the incident waves. Since the media above and below the multilayer are isotropic, we may define

$$h_d^{(0)} = h_1^{(0)} = h_2^{(0)} \qquad h_u^{(0)} = h_3^{(0)} = h_4^{(0)}$$

$$h_d^{(N+1)} = h_1^{(N+1)} = h_2^{(N+1)} \qquad h_u^{(N+1)} = h_3^{(N+1)} = h_4^{(N+1)}$$

and

$$\bar{E}_{in} = (c_3^{(0)}\bar{A}_3^{(0)} + c_4^{(0)}\bar{A}_4^{(0)})e^{i(-\omega t + kx + h_u^{(0)}z)} \qquad \text{for } z \le 0$$

$$(21b)\qquad \bar{E}_{ref} = (c_1^{(0)}\bar{A}_1^{(0)} + c_2^{(0)}\bar{A}_2^{(0)})e^{i(-\omega t + kx + h_d^{(0)}z)} \qquad \text{for } z \le 0$$

$$\bar{E}_{tr} = (c_3^{(N+1)}\bar{A}_3^{(N+1)} + c_4^{(N+1)}\bar{A}_4^{(N+1)})e^{i(-\omega t + kx + h_u^{(N+1)}z)} \qquad \text{for } z \ge L$$

So that equations (21a) yield, as our final result, the connection between the incident field $\bar{E}_{in}$ and the reflected and transmitted fields $\bar{E}_{ref}$ and $\bar{E}_{tr}$.

D - Application to the Ionosphere

In section C, the coupling and scattering matrices were derived for an arbitrary anisotropic multilayer subject only to the restriction that μ was a scalar. In practice, one of the most important anisotropic media considered is the ionosphere, in which the conductivity tensor $\bar{\bar{\sigma}}$ is due to free electrons in a damping medium permeated by a fixed uniform magnetic field and in which μ and ϵ are scalar constants. $\bar{\bar{\sigma}}$ is found by solving, first, for the velocity $\bar{v}$ from the equation of motion (in M.K.S. units)

$$m\dot{\bar{v}} + m\nu\bar{v} = -e\bar{E}_o e^{-i\omega t} - \mu e\,\bar{v} \times \bar{H}$$

of an electron in a resistive medium under the influence of a fixed magnetic field $\bar{H}$ and an oscillating electric field $\bar{E}_o e^{-i\omega t}$. Here m is the mass of the electron, ν is the collision frequency and e is the magnitude of the electronic charge (the sign having already been taken into account).

Using the relation

$$\bar{\bar{\sigma}}\bar{E}_o e^{-i\omega t} = \bar{j} = -Ne\bar{v}$$

for the current density $\bar{j}$ in terms of the impressed electric field $\bar{E}_o e^{-i\omega t}$ where N is the number of electrons per unit volume, $\bar{\bar{\sigma}}$ can be obtained from $\bar{v}$. Letting ω, ω_H and ω_{Cr} represent the angular velocities associated respectively with the frequency of the incident wave, the gyromagnetic frequency and the critical frequency, i.e.

$$\omega_H = \frac{\mu e\,|\bar{H}|}{m}\,,\quad \omega_{Cr} = \sqrt{\frac{e^2N}{\epsilon m}}\quad ; \tag{22}$$

and letting α, β, γ be the direction cosines of the magnetic field $\bar{H}$, $\bar{\bar{\sigma}}$ is given by

$$\bar{\bar{\sigma}} = \frac{i\epsilon\omega_{Cr}^2}{(\omega+i\nu)\left[(\omega+i\nu)^2-\omega_H^2\right]} \begin{Vmatrix} (\omega+i\nu)^2-\omega_H^2\alpha^2 & -i(\omega+i\nu)\omega_H\gamma - \omega_H^2\alpha\beta & i(\omega+i\nu)\omega_H\beta-\omega_H^2\alpha\gamma \\ i(\omega+i\nu)\omega_H\gamma-\omega_H^2\beta\alpha & (\omega+i\nu)^2-\omega_H^2\beta^2 & -i(\omega+i\nu)\omega_H\alpha-\omega_H^2\beta\gamma \\ -i(\omega+i\nu)\omega_H\beta-\omega_H^2\gamma\alpha & i(\omega+i\nu)\omega_H\alpha-\omega_H^2\gamma\beta & (\omega+i\nu)^2-\omega_H^2\gamma^2 \end{Vmatrix} \tag{23}$$

With $\bar{\bar{\sigma}}$ given, $\bar{\bar{\rho}}$ is determined by equation (4b):

$$\bar{\bar{\rho}} = \omega^2 \mu\epsilon \left(\bar{\bar{I}} + \frac{1}{\omega\epsilon}\,\bar{\bar{\sigma}}\right) \tag{24}$$

for scalar ϵ. From $\bar{\bar{\rho}}$, the scattering matrix for the ionospheric multilayer can be computed.

That the multilayer can be used to approximate a continuously varying medium follows from the fact the scattering matrix is a continuous functional of the charge density and collision frequency. It is a consequence of this fact that as the number of layers in the multilayer tends toward infinity so that the multilayer approaches a continuous medium, the scattering matrix for the multilayer approaches the scattering matrix for that continuous medium. Hence we are led to expect that for a reasonably large number of thin layers, the scattering matrix for the multilayer will be a good approximation to the scattering matrix for the continuous medium it represents. This will be the case if each layer is but a small fraction of a wavelength in thickness.

3. THE CONTINUOUSLY VARYING MEDIUM AT HIGH FREQUENCIES

The multilayer provides an approximation to the actual scattering matrix for the continuous medium in the low frequency range which cannot, in general, be treated by other methods. In the high frequency range, however, a solution in series form exists which is essentially of the W.K.B. type. The n^{th} term in the series is of the order of ω^{-2n} as $\omega \longrightarrow \infty$, as will be seen subsequently. As a result, it is possible to effect a considerable simplification in the analysis by expanding the quantities appearing in each term in the series in inverse powers of ω. The importance of such a simplification can be appreciated when it is realized that the quantities appearing in the original series involve the solutions of a quartic equation and various complicated functions of these solutions. On the other hand, by virtue of the expansion in inverse powers of ω it is possible to obtain relatively simple useable series expansions for the total coupling and scattering matrices as explicit functions of all ionospheric parameters.

Maxwell's equations can be transformed into a system of four coupled, linear first order differential equations* in four amplitude functions $V_n(z)$ by substituting

$$\bar{E}(x,z,t) = \sum_{n=1}^{4} V_n(z)\bar{A}_n(z)e^{i\left[-\omega t+kx+\int_0^z h_n(\zeta)d\zeta\right]}$$

(25)

$$\bar{H}(x,z,t) = \sum_{n=1}^{4} V_n(z)\bar{D}_n(z)e^{i\left[-\omega t+kx+\int_0^z h_n(\zeta)d\zeta\right]}$$

into Maxwell's equations, where $\bar{A}_n$, $\bar{D}_n$ and h_n satisfy the algebraic relations

(26)

(a) $-h_n A_{ny} = -\omega\mu D_{nx}$

(b) $h_n A_{nx} - kA_{nz} = -\omega\mu D_{ny}$

(c) $kA_{ny} = -\omega\mu D_{nz}$

(d) $-h_n D_{ny} = \frac{1}{\omega\mu}(\bar{\bar{\rho}}\bar{A}_n)_x$

(e) $h_n D_{nx} - kD_{nz} = \frac{1}{\omega\mu}(\bar{\bar{\rho}}\bar{A}_n)_y$

(f) $kD_{ny} = \frac{1}{\omega\mu}(\bar{\bar{\rho}}\bar{A}_n)_z$

which are equivalent to Maxwell's equations for a plane wave

$$\bar{E}_n = \bar{A}_n\, e^{i\left[-\omega t+k_x+h_n z\right]}$$

(27)

$$\bar{H}_n = \bar{D}_n\, e^{i\left[-\omega t+k_x+h_n z\right]}$$

---

*H. Bremmer, Loc. cit.

traveling in a homogeneous medium whose properties are the same as those of the actual medium at the point z. This substitution yields, for n = 1,2,3,4, the differential equations satisfied by $V_n(z)$

$$(28) \qquad \frac{dV_n}{dz} = -e^{-i\int_o^z h_n(\zeta)d\zeta} \sum_{m=1}^{4} \alpha_{nm}(z)\, V_m(z) e^{i\int_o^z h_m(\zeta)d\zeta}$$

where

$$(29a) \qquad \alpha_{nm} = \frac{\Delta_{nm}}{\Delta}$$

$$(29b) \qquad \Delta = \begin{Vmatrix} A_{1x}, & A_{2x}, & A_{3x}, & A_{4x} \\ A_{1y}, & A_{2y}, & A_{3y}, & A_{4y} \\ h_1A_{1y}, & h_2A_{2y}, & h_3A_{3y}, & h_4A_{4y} \\ B_1, & B_2, & B_3, & B_4 \end{Vmatrix}$$

and Δ_{nm} is equal to Δ with the n^{th} column removed and replaced by the column $\frac{dA_{mx}}{dz}$, $\frac{dA_{my}}{dz}$, $\frac{d}{dz}(h_mA_{my})$, $\frac{dB_m}{dz}$ with

$$(29c) \qquad B_n = h_nA_{nx} - kA_{nz}$$

Thus, for example

$$(29d) \qquad \alpha_{23} = \frac{1}{\Delta} \begin{Vmatrix} A_{1x}, & \frac{dA_{3x}}{dz}, & A_{3x}, & A_{4x} \\ A_{1y}, & \frac{dA_{3y}}{dz}, & A_{3y}, & A_{4y} \\ h_1A_{1y}, & \frac{d}{dz}(h_3A_{3y}), & h_3A_{3y}, & h_4A_{4y} \\ B_1, & \frac{dB_3}{dz}, & B_3, & B_4 \end{Vmatrix}$$

The α_{nm} given in (29) are exactly the same as those given by Bremmer. However, they are here written only in terms of the electric field, the tangential components of the magnetic field D_{nx} and D_{ny} having been eliminated with the help of (26a and b).

The solution to (28) is

$$(30) \qquad V_n(z) = \sum_{m=1}^{4} M_{nm}(z)\, V_m(0)$$

where $M_{nm}(z)$ is given by the series

$$(31)\quad M_{nm}(z) = \sum_{j=o}^{\infty} M_{nm}^{j}(z) \quad , \text{ where}$$

$$M_{nm}^{(0)}(z) = \delta_{nm}$$

$$M_{nm}^{(1)}(z) = -e^{-i\int_o^z h_n(\zeta)d\zeta} \int_o^z dz_1 e^{i\left[\int_{z_1}^z h_n(\zeta)d\zeta + \int_o^{z_1} h_m(\zeta)d\zeta\right]} \alpha_{nm}(z_1)$$

$$\cdots$$

$$M_{nm}^{(j)}(z) = (-1)^j e^{-i\int_o^z h_n(\zeta)d\zeta} \cdot \sum_{s_1=1}^{4} \cdots \sum_{s_{j-1}=1}^{4} \int_o^z dz_1 \int_o^{z_1} dz_2 \cdots \int_o^{z_{j-1}} dz_j U$$

(32)

$$U = \alpha_{ns_1}(z_1)\alpha_{s_1s_2}(z_2)\cdots\alpha_{s_{j-1}m}(z_j)$$

$$\cdot \exp i\left[\int_{z_1}^z h_n(\zeta)d\zeta + \sum_{k=1}^{j-1} \int_{z_{k+1}}^{z_k} h_{s_k}(\zeta)d\zeta + \int_o^{z_j} h_m(\zeta)d\zeta\right] \quad \text{for } j \geq 1$$

That (30) is formally a solution of (28) can be verified by direct substitution.

The series (31) for $M_{nm}(z)$ is given in terms of the h_r and the α_{rs}. The former are the solutions of a quartic equation; the components of the matrix $\bar{\bar{\alpha}}$ are given in (29) as the ratio of two four by four determinants whose elements, in turn, are functions of the h_r. It is thus evident that the formal solution (30) is rather impractical for demonstrating the overall dependence of the scattering matrix on the ionospheric properties.

On the other hand, if $\bar{\bar{\rho}}$ is expanded in terms of inverse powers of the frequency, h_n and $\bar{\bar{\alpha}}$ can also be obtained as series expansions. It is then possible to obtain relatively simple, useable series expansions for the total coupling and scattering matrices as explicit functions of all ionospheric paramters. These expansions will converge at frequencies greater than both the gyromagnetic and critical frequencies, which is exactly the frequency range in which (31) is rapidly convergent.

Thus, writing $\bar{\bar{\sigma}}$ as a power series in the dimensionless parameter $\lambda = \omega_H/\omega$, we have

$$(33)\quad \bar{\bar{\rho}} = \frac{\omega_H^2}{c^2\lambda^2}\left(\bar{\bar{I}} + \frac{i\lambda}{\omega_H \epsilon} \sum_{r=o}^{\infty} \bar{\bar{\sigma}}(r)\lambda^r\right)$$

$\bar{\bar{I}}$ being the unit matrix. The first few terms of this expansion are

$$\bar{\bar{\rho}} = \frac{\omega_H^2}{c^2\lambda^2}\,\bar{\bar{1}} - \frac{\omega_{Cr}^2}{c^2}\,\bar{\bar{1}} + \lambda i\,\frac{\omega_{Cr}^2}{c^2}\begin{Vmatrix} +a & ; & +\gamma & ; & -\beta \\ -\gamma & ; & +a & ; & +\alpha \\ +\beta & ; & -\alpha & ; & +a \end{Vmatrix}$$

(34)

$$+\lambda^2\,\frac{\omega_{Cr}^2}{c^2}\begin{Vmatrix} a^2+\alpha^2-1 & ; & \beta\alpha+2a\gamma & ; & \gamma\alpha-2a\beta \\ \alpha\beta-2a\gamma & ; & a^2+\beta^2-1 & ; & \gamma\beta+2a\alpha \\ \alpha\gamma+2a\beta & ; & \beta\gamma-2a\alpha & ; & a^2+\gamma^2-1 \end{Vmatrix}$$

$$+\lambda^3 i\,\frac{\omega_{Cr}^2}{c^2}\begin{Vmatrix} a(3-3\alpha^2-a^2) & ; & \gamma-3a^2\gamma-3a\beta\alpha & ; & -\beta+3a^2\beta-3a\gamma\alpha \\ -\gamma+3a^2\gamma-3a\alpha\beta & ; & a(3-3\beta^2-a^2) & ; & \alpha-3a^2\alpha-3a\gamma\beta \\ \beta-3a^2\beta-3a\alpha\gamma & ; & -\alpha+3a^2\alpha-3a\beta\gamma & ; & a(3-3\gamma^2-a^2) \end{Vmatrix} + + + \ .$$

where a is defined by $a = \frac{\nu}{\omega_H}$. Substituting this expression into the three equations (5), we can solve for h_n and $\bar{A}_n$ as series expansions in λ:

(35) a) $$h_n = -\xi_n \frac{\omega}{C}\cos\theta + \sum_{r=0}^{\infty} h_n^{(r)}\lambda^r(r) = -\xi_n \frac{\omega_H}{C\lambda}\cos\theta + \sum_{r=0}^{\infty} h_n^{(r)}\lambda^r$$

b) $$\bar{A}_n = \sum_{r=0}^{\infty} \bar{A}_n^{(r)}\lambda^r$$

$\xi_1 = \xi_2 = 1$ and $\xi_3 = \xi_4 = -1$ so that the term $-\xi_n \frac{\omega}{C}\cos\theta$ in (35) expresses the fact that as $\omega \longrightarrow \infty$ and hence $\lambda \longrightarrow 0$, $h_n \longrightarrow -\xi_n \frac{\omega}{C}\cos\theta$, the values of the z-component of the propagation vector in free space for the downcoming and upgoing waves respectively.

Carrying out the indicated substitutions and subsequently equating the respective coefficients of λ^r to zero in each of the three equations (5) yields, for each r, a set of three simultaneous equations which can be solved for $h_n^{(r+1)}$, a relation between $A_{nx}^{(r+2)}$ and $A_{ny}^{(r+2)}$, and finally, for $r \geq 1$, a relation between the three components of $\bar{A}_n^{(r-1)}$. Thus, after a little collecting, we find that there are two equations connecting the three components of $\bar{A}_n^{(r)}$, while $h_n^{(r)}$ is uniquely determined. This corresponds to the fact that equations (5) determine the h_n uniquely, while yielding only two relations between the three components of $\bar{A}_n$. (i.e. the normalization of the polarization vectors $\bar{A}_n$ remains arbitrary.) Although, at first sight, it might seem strange that the two equations connecting the components of $\bar{A}_n^{(j)}$ are obtained from different orders in the expansion of equations (5), a little consideration makes it clear that this lack of symmetry is necessitated by the requirement that if the frequency of the incident wave becomes infinite or if the charge density vanishes, the polarization vectors $\bar{A}_n$ must become perpendicular to the propagation vectors. That is, $kA_{nx} + h_n A_{nz} = 0$ if $\lambda = 0$,

or if N = 0. Thus, the equation connecting $A_{nx}^{(r)}$ and $A_{nz}^{(r)}$ stems from a different source than the other equation connecting the three components of $\bar{A}_n^{(r)}$, and we must not be surprised to find this lack of symmetry.

The expansions for h_n and $\bar{A}_n$ have been evaluated to the order λ^2

$$h_n = -\frac{1}{\lambda}\xi_n \frac{\omega_H}{C}\cos\theta + \lambda\xi_n \frac{\omega_{Cr}^2}{2\omega_H C\cos\theta} - \lambda^2\chi_n \frac{\omega_{Cr}^2}{2\omega_H C\cos\theta}\left(\cos\phi_n + i\zeta_n\frac{\nu}{\omega_H}\right)$$

$$A_{nx} = \xi_n\cos\theta - \lambda^2\xi_n \frac{\omega_{Cr}^2}{2\omega_H^2\cos\theta}$$

(36)

$$A_{ny} = i\zeta_n - \lambda\frac{i}{2}\frac{a_n^2}{\cos\phi_n} + \lambda^2\frac{i\zeta_n}{2}\left\{+\frac{a_n^4}{4\cos^2\phi_n} + i\zeta_n\frac{\nu}{\omega_H}\frac{a_n^2}{\cos\phi_n} - \frac{\omega_{Cr}^2}{\omega_H^2}\right\}$$

$$A_{nz} = \sin\theta + 0\lambda + 0\lambda^2$$

where we have so normalized $\bar{A}_n$ that $A_{nz}^{(1)} = A_{nz}^{(2)} = 0$. The notation used above is summarized on Table I.

$$\xi_1 = 1 \quad \xi_2 = 1 \quad \xi_3 = -1 \quad \xi_4 = -1$$

$$\chi_1 = 1 \quad \chi_2 = -1 \quad \chi_3 = 1 \quad \chi_4 = -1$$

$$\zeta_1 = 1 \quad \zeta_2 = -1 \quad \zeta_3 = -1 \quad \zeta_4 = 1$$

$$\lambda = \frac{\omega_H}{\omega}, \omega_{Cr}(z) = \sqrt{\frac{e^2N(z)}{\epsilon m}}, \quad \nu(z) \text{ is the collision frequency,}$$

$$a_n = \alpha\xi_n\cos\theta + \beta i\zeta_n + \gamma\sin\theta = \frac{\bar{H}\cdot\bar{A}_n^{(0)}}{|H|}$$

$$\cos\phi_n = \alpha\sin\theta - \gamma\xi_n\cos\theta$$

TABLE I

It will be noticed that ϕ_n is the angle made by the incident wave normal of the n^{th} mode with the direction of the magnetic field.

The expansions for $h_n A_{ny}$ and B_n are simply obtained from those given by (36). They are, to order λ^2 :

$$h_n A_{ny} = -\frac{1}{\lambda} i\chi_n \frac{\omega_H}{C}\cos\theta - i\xi_n \frac{\omega_H a_n^2\cos\theta}{2C\cos\phi_n}$$

$$- \lambda i\chi_n \frac{\omega_H\cos\theta}{2C}\left\{\frac{a_n^4}{4\cos^2\phi_n} + i\zeta_n\frac{\nu}{\omega_H}\frac{a_n^2}{\cos\phi_n} - \frac{\omega_{Cr}^2}{\omega_H^2}\frac{1+\cos^2\theta}{\cos^2\theta}\right\}$$

(37)

$$B_n = h_n A_{nx} - kA_{nz} = -\frac{1}{\lambda}\frac{\omega_H}{C} + \lambda\frac{\omega_{Cr}^2}{\omega_H C}$$

Having, now, the expansions for A_{nx}, A_{ny}, $h_n A_{ny}$ and B_n, we can compute the expansions for Δ_{nm}, Δ, and hence α_{nm}. Owing to the fact that the first two terms in the expansions for A_{mx}, A_{my}, $h_m A_{my}$ and B_m are independent of z, the series representing α_{nm} will begin with the term in λ^2, so that

$$(38)\qquad \alpha_{nm}(z) = \sum_{r=2}^{\infty} \alpha_{nm}^{(r)}(z)\lambda^r$$

Denoting the exponential phase factor for the jth term in (32) by $\Phi^{(j)}_{n,s_1,\ldots,s_{j-1},m}(z,z_1,\ldots,z_j)$:

$$(39)\qquad \Phi_{nm}^{(o)} = 1$$

$$\Phi_{nm}^{(1)} = e^{i\left[-\int_o^{z_1} h_n(\zeta)d\zeta + \int_o^{z_1} h_m(\zeta)d\zeta\right]}$$

$$\Phi^{(j)}_{n,s_1,\ldots,s_{j-1},m} = e^{i\left[-\int_o^{z_1} h_n(\zeta)d\zeta + \sum_{k=1}^{j-1} h_{s_k}(\zeta)d\zeta + \int_o^{z_j} h_m(\zeta)d\zeta\right]}$$

the subscripts $n, s_1, \ldots, s_{j-1}, m$ taking on the values 1,2,3,4, we have for $M_{nm}^{(j)}(z)$

$$(40)\qquad M_{nm}^{(0)}(z) = \delta_{nm}$$

$$M_{nm}^{(1)}(z) = -\int_o^z dz_1 \Phi_{nm}(z,z_1) \sum_{r=2}^{\infty} \alpha_{nm}^{(r)}(z_1)\lambda^{(r)}$$

$$\cdots$$

$$M_{nm}^{(j)}(z) = (-1)^j \sum_{s_1=1}^{4} \cdots \sum_{s_{j-1}=1}^{4} \int_o^z dz_1 \int_o^{z_1} dz_2 \cdots \int_o^{z_{j-1}} dz_j \Phi_{n_1 s_1,\ldots,s_{j-1},m}(z,z_1,\ldots$$

$$\left(\sum_{r=2}^{\infty} \alpha_{ns_1}^{(r)}(z_1)\lambda^r\right)\left(\sum_{r=2}^{\infty} \alpha_{s_1 s_2}^{(r)}(z_2)\lambda^r\right)\cdots\left(\sum_{r=2}^{\infty} \alpha_{s_{j-1}m}^{(r)}(z_j)\lambda^r\right)$$

In (40) we have not expanded the exponential phase factor in powers of λ, because the coefficients in such an expansion, involving integrals of the propagation vector, will be roughly proportional to the thickness of the ionosphere in wavelengths. At high frequencies, this will be very large, so that the coefficient of λ^r in the expansions of the Φ's will be very much larger than the coefficient of λ^r in the expansion of $\bar{\bar{\alpha}}$. Leaving the exponential unexpanded will, on the other hand, insure that terms of different order in λ, in the expansion of $M_{nm}(z)$, will be of different order of magnitude in cases of practical interest.

It can be seen from (40) that $M_{nm}^{(j)}$ involves the product of j $\bar{\bar{\alpha}}$'s; thus, since the series expansion for $\bar{\bar{\alpha}}$ begins with the terms in λ^2, the expansion for $M_{nm}^{(j)}$ begins with the term in λ^{2j} as mentioned earlier. Consequently, the contribution due to multiple reflections decreases with increasing frequency.

The leading term $\alpha_{nm}^{(2)}(z)$ in the expansion of α_{nm} has been calculated:

$$(41)\quad \alpha_{nm}^{(2)}(z) = -\frac{(\omega_{Cr}^2)'}{4\omega_H^2}\left(1+\frac{\gamma_n\gamma_m+\zeta_n\zeta_m}{2}+\frac{\xi_n\xi_m+\gamma_n\gamma_m}{2\cos^2\theta}\right)+\nu'\frac{ia_n^2}{8\omega_H\cos\phi_m}$$

where ω_{Cr} and ν are functions of z and the prime denotes differentiation. From this we obtain $M_{nm}(z)$ to order λ^2 by adding the contributions due to $M_{nm}^{(0)}(z)$ and $M_{nm}^{(1)}$ which are the only terms that contribute to order λ^2

$$(42)\quad M_{nm}(z) = \delta_{nm}-\lambda^2\int_o^z dz_1\, e^{i\int_o^{z_1}\left[-h_n(\zeta)+h_m(\zeta)\right]d\zeta}\alpha_{nm}^{(2)}(z_1)\quad .$$

In (42) the factor $e^{-i\int_o^z h_n(\zeta)d\zeta}$ was taken into the integral

$$e^{-i\int_o^z h_n(\zeta)d\zeta}\cdot e^{i\left[\int_{z_1}^z h_n(\zeta)d\zeta+\int_o^{z_1}h_m(\zeta)d\zeta\right]} = e^{i\int_o^{z_1}\left[-h_n(\zeta)+h_m(\zeta)\right]d\zeta}.$$

The scattering matrix for a continously varying anisotropic medium of thickness L is now obtainable (to order λ^2) from $M_{nm}(L)$. Defining the amplitudes $C_n(z)$ by the relation

$$(43)\qquad C_n(z)e^{ih_n(z)z} = V_n(z)e^{i\int_o^z h_n(\zeta)d\zeta},$$

we have, from (25),

$$(44)\qquad \bar{E}(x,z,t) = \sum_{n=1}^{4} C_n(z)\bar{A}_n(z)e^{i\left[-\omega t+kx+h_n(z)z\right]}\quad .$$

The coupling matrix $\bar{\bar{T}}$ for the medium is defined by

$$(45)\qquad C_n(L) = \sum_{m=1}^{4} T_{nm}C_m(0)\quad .$$

Substitution of (43) into (30) and setting z = L yields

$$(46)\quad T_{nm} = e^{-ih_n(L)L}\,e^{i\int_o^L h_n(\zeta)d\zeta}\,M_{nm}(L)$$

The scattering matrix $\bar{\bar{S}}$ is given in terms of the elements of $\bar{\bar{T}}$ by equations (19b) and (20). If the incoming field is incident from below, the reflected and transmitted fields are given by equations (21). The components of $\bar{\bar{T}}$ to order λ^2 are readily obtained by substituting (42), (41) and the first of equations (36) into (46).

No special cases will be considered here but it can be concluded in general from the above results that there is an anomalous effect when the direction of either an incident or reflected wave coincides with the direction of the earth's magnetic field. Attenuation in the medium is roughly proportional to the path length of a geometrical optics ray in the medium and depends on the product of the collision frequency and charge density. Strictly speaking, the absorption factor, to order λ^2, is just the absorption that would be suffered by a geometrical optics ray which is incident at angle θ and traverses the medium undeviated except for a transition, at the height z_1, from the m^{th} mode to the n^{th}, provided that the m^{th} mode is upgoing and the n^{th} is downcoming. Higher order terms take into consideration the curvature of the path in the medium and the effect of the earth's magnetic field.

From (41), it is seen that even aside from the attenuation effects just mentioned, the scattering matrix will depend on the collision frequency through the term

$$\lambda^2 \nu' \frac{ia_n^2}{8\omega_H \cos\phi_m}$$

and that this effect will depend also upon the magnitude of the earth's magnetic field and the direction of the incident wave with respect to this field. Carrying out the expansions to higher order will, of course, yield better accuracy. However, the major effects already appear in the λ^2 term in the frequency expansion.

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